Journal of Mathematical Sciences: Advances and Applications
Volume 22, 2013, Pages 45-52

A PROOF FOR GOLDBACH'S CONJECTURE

HASHEM SAZEGAR

Department of Mathematics
Azad Mashhad University
Mashhad
Iran
e-mail: h.sazegar@gmail.com

Abstract

In 1742, Goldbach claimed that each even number can be shown by two primes. In 1937, Vinograd of Russian Mathematician proved that each odd large number can be shown by three primes. In 1930, Lev Schnirelmann proved that each natural number can be shown by M-primes. In 1973, Chen Jingrun proved that each odd number can be shown by one prime plus a number that has maximum two primes. In this article, we state one proof for Goldbach's conjecture.

1. Introduction

Bertrand's postulate state for every positive integer n, there is always at least one prime p, such that $n < p < 2n$. This was first proved by Chebyshev in 1850, which is why postulate is also called the Bertrand-Chebyshev theorem.

2010 Mathematics Subject Classification: Primary 11P32; Secondary 11A41.

Keywords and phrases: Bertrand-Chebyshev theorem, Landau's problems, twin prime, Legendre's conjecture, Oppermann's conjecture.

Received November 14, 2012

Legendre's conjecture states that there is a prime between n^2 and $(n+1)^2$ for every positive integer n, which is one of the four Landau's problems. The rest of these four basic problems are

(i) Twin prime conjecture: There are infinitely many primes p such that $p+2$ is a prime.

(ii) Goldbach's conjecture: Every even integer $n > 2$ can be written as the sum of two primes.

(iii) Are there infinitely many primes p such that $p-1$ is a perfect square? Problems (i), (ii), and (iii) are open till date.

Legendre's conjecture is proved (Sazegar [8]).

In 1742, Goldbach claimed that each even number can be written as the sum of two primes. In 1930, Lev Schnirelmann proved that every even number $n \geq 4$ can be written as the sum of at most 20 primes. This result was subsequently enhanced by many authors; currently, the best known result is due to Olivier Ramar, who in 1995 showed that every even number $n \geq 4$ is in fact the sum of at most six primes. In fact, resolving the weak Goldbach conjecture will also directly imply that every even number $n \geq 4$ is the sum of at most four primes (Sinisalo and Matti [9]). In 1937, Vinogradoff of Russian Mathematician proved that each odd large number can be written as the sum of three primes. Chen Jingrun showed in 1973 using the methods of Sieve theory that every sufficiently large even number can be written as the sum of either two primes, or a prime and a semiprime, (the product of two primes) (Chen [10]) e.g., $100 = 23 + 7 \times 11$.

In 1975, Hugh Montgomery and Robert Charles Vaughan showed that "most" even numbers were expressible as the sum of two primes. More precisely, they showed that there existed positive constants c and C such that for all sufficiently large numbers N, every even number less than N is the sum of two primes, with at most CN^{1-c} exceptions. In particular, the set of even integers, which are not the sum of two primes has density zero.

Linnik proved in 1951, the existence of a constant K such that every sufficiently large even number is the sum of two primes and at most K powers of 2. Roger Heath-Brown and Jan-Christoph Schlage-Puchta in 2002 found that $K = 13$ works (Heath-Brown and Puchta [11]). This was improved to $K = 8$ by Pintz and Ruzsa in 2003 (Pintz and Ruzsa [12]).

Theorem. *Every even large positive integer can be written as the sum of two primes.*

Before solving this theorem, we state some equations as follow:

$$N - p_1 = m_1,$$

$$N - p_2 = m_2,$$

$$\vdots$$

$$N - p_r = m_r.$$

That $p_1 = 3,\ p_2 = 5, \dots.$

Notice that p_r is a large prime such that $N^{2/3} \leq p_r \leq 2N^{2/3}$, notice that there is a prime in this interval, N is a large even number, and $m_1, \dots, m_r$ are composite odd numbers, otherwise our theorem is correct.

We prove this theorem by induction and we assume that all $m_1, \dots, m_r$ are composite and we reach to a contradiction.

To proceed to this proof, firstly, we use the following lemmas:

2. Lemmas

In this section, we present several lemmas, which are used in the proof of our main theorem.

Before state the below lemmas, we write $m_1 \dots m_r \leq m_1.a_1.a_2 \dots a_{r-1}$, that $a_1, \dots, a_{r-1}$ are composite odd numbers, less than m_1, and may be equal or larger than $m_2, \dots, m_r$ for simplicity, we denote $m_1 = a_0$.

Notice that in below lemmas, we state the properties of these numbers.

Lemma 2.1. *If the number of composite odd numbers to be* r *so the number of total odd numbers* (*composite and prime*) *are at most* $9r/4$, *i.e.,* a_0 *is at most* $a_{r-1} + 9r/2$.

Proof. Let p is a large prime number, we know from odd numbers

$$p,\ p+2,\ p+4,\ p+6,\ p+8,\ p+10,\ p+12,\ p+14,\ p+16,$$

numbers $p+4,\ p+10,\ p+16$ are composite (we assume that $p+2$ is prime), since obviously that $p+4,\ p+10,\ p+16$ are composite, but if $p = 5k+1$ so $p+14$ is composite, if $p = 5k+2$ so $p+8$ is composite, if $p = 5k+4$ so $p+6$ is composite, if $p = 5k+3$ so $p+2$ is composite but we assume that $p+2$ is prime, so the number of composite odd numbers is 4, therefore if the number of composite odd numbers to be r, the number of total odd numbers is $r + 5r/4 = 9r/4$, so the total number is at most $9r/2$.

Notice that with any method, the number of prime numbers is at most 5.

Lemma 2.2. *If* $N^{2/3} \leq p_r \leq 2N^{2/3}$, *then for a large* r, N, *we have* $r \geq N^{20/33}$.

Proof. According to (Hardy and Wright [3]), there are constant numbers c_1 and c_2 such that $c_1 r \log r \leq p_r \leq c_2 r \log r$, then $N^{2/3} \leq p_r \leq r^{11/10}$ so $r \geq N^{20/33}$.

Lemma 2.3. *All prime factors* q, *where* $3 \leq q \leq \sqrt{N}$ *appeared in numbers* $a_0, a_1, a_2, \ldots, a_{r-1}$.

Proof. Assume that there is a prime number like q' has not appeared in $a_0, a_1, a_2 \ldots, a_{r-1,}$ so we can write $a_0 = q't + f$ that $0 < f \leq q' - 1 < \sqrt{N}$ but since $r \geq N^{20/33}$ and $3 \leq q' \leq \sqrt{N}$ so $a_0 - f$

or $a_0 - f - q'$ is one of numbers $a_1, a_2 \dots, a_{r-1}$ because $a_{r-1} < a_0 - \sqrt{N} < a_0 - f < a_0$ or $a_{r-1} < a_0 - 2\sqrt{N} < a_0 - f - q' < a_0$ we know $a_0 - a_r >= 2r >= 2N^{(20/33)} >= 2N^{(1/2)}$.

Lemma 2.4. *Let l denote the number of $3 \le q \le \sqrt{a_0}$, are in odd composite numbers $a_0, a_1, a_2, \dots, a_{r-1}$ so $l \le \frac{9r}{4q}$, for all $3 \le q \le \sqrt{a_0}$.*

Proof. If $q \ge 3$, we have $a_i - a_j = 2ql \le \frac{9r}{2}$, so $l \le \frac{9r}{4q}$, that l is the number of $q \ge 3$ in numbers $a_0, a_1, a_2, \dots, a_{r-1}$, which are composite odd numbers.

Lemma 2.5. *Let p is prime and f denote the number of $p \ge \sqrt{a_i}$, in which $0 \le i \le r-1$ are in $a_0, a_1, a_2, \dots, a_{r-1}$, i.e., $a_i = tp$, that these numbers are odd composite numbers.*

So the number of such p are

$$f \le \frac{9r/4}{9},$$

or

$$f \le \frac{9r/4(1-1/9)}{3 \times 5},$$

or

$$f \le \frac{9r/4(1-1/9-1/15)}{3 \times 7},$$

.

.

.

we continue this method to reach $1 - 1/9 - 1/15 - 1/21 - \dots - 1/3 \times 89 =$ almost 0.56, because we have r odd composite numbers.

Proof. If $p \geq \sqrt{a_i}$, in which $0 \leq i \leq r-1$, that $a_i = qp$, so $a_{r-1}/q \leq p \leq a_0/q$, in which $3 \leq q \leq \sqrt{a_0}$. Since the distance of between two odd numbers should be 2, so $a_{r-1}/q - 2 \leq p - 2 = p' \leq a_0/q - 2$ that p' is prime but in $a_{r-1}/q - 4 \leq p - 4 = w \leq a_0/q - 4, w$ is not prime, so if $q = 3$, the number of such p is

$$f \leq \frac{9r/2}{2 \times 3} \times \frac{2}{3} \times \frac{1}{2} = \frac{9r/4}{9}.$$

But since $p \geq \sqrt{a_i}$, only one $p \geq \sqrt{a_i}$, in which $0 \leq i \leq r-1$, could be in $a_0, a_1, a_2, \ldots, a_{r-1}$, so for $q = 5$,

$$f \leq \frac{9r/4(1 - 1/9)}{3 \times 5}.$$

For $q = 7$,

$$f \leq \frac{9r/4(1 - 1/9 - 1/15)}{3 \times 7},$$

we continue this method to reach $1 - 1/9 - 1/15 - 1/21 - \ldots - 1/267 =$ *almost* 0.56.

Note. We have only $9r/4$ odd numbers, since we say about $p \geq \sqrt{a_i}$ (this is new idea) not old idea, i.e., $q \leq \sqrt{a_i}$, for $q = 3$, we have $(9r/4)/9$ such $p \geq \sqrt{a_i}$, since we have only one such $p \geq \sqrt{a_i}$, exist, if we have two such primes, i.e., $p_1 p_2 q > a_i$ and this is contradiction, so for $q = 5$ $9r/4$ numbers changed to $(9r/4) - (9r/4)/9$, for $q = 7$, these numbers changed to $9r/4 - (9r/4)/9 - (9r/4)/15$, we continue this method to reach $9r/4 - (9r/4)/9 - (9r/4)/15 - \ldots - (9r/4)/3 \times 89 =$ almost 0.56, since we have only r composite numbers.

Lemma 2.6. *We always have*

$$\log(N-3)\ldots(N-p_r) > r\log(N-p_r).$$

Proof. Since all numbers $N - 3, \ldots, N - p_{r-1}$ are larger than $N - p_r$, this result is obvious.

3. The Proof of Main Theorem

Theorem. *Every even large positive integer can be written as the sum of two primes.*

Proof. Let all even numbers smaller than N be represented by two primes by induction, and we assume that all $a_0, a_1, a_2, \ldots, a_{r-1}$ are composite odd numbers, and we reach to a contradiction.

According to (Hardy and Wright [3]), there is a prime factor like q that for any composite odd numbers $a_0, a_1, a_2, \ldots,\ a_{r-1}, q \leq \sqrt{a_i}$, in which $0 \leq i \leq r-1$.

Now, we use the above results to reach a contradiction. According to Lemma 2.4,

$$\log(N-3)\ldots(N-p_r) > r\log(N-p_r).$$

According to Lemmas 2.4 and 2.5, and for a large r and N, we have

$$(N-3)\ldots(N-p_r) \leq 3^{\frac{9r/4}{3}} \times \ldots \times 89^{\frac{9r/4}{89}} \times \frac{N}{3}^{\frac{9r/4}{9}} \times \frac{N}{5}^{\frac{9r/4(1-1/9)}{3\times 5}} \times \ldots .$$

We continue to reach $1 - 1/9 - 1/15 - 1/21 - \ldots - 1/267 =$ almost 0.56. Hence, we have

$$r\log(N-p_r) < 9r/4\sum_{3\leq q\leq 89}\frac{\log q}{q} + 9r/4\left(\frac{1}{9} + \frac{(1-1/9)}{3\times 5} + \frac{(1-1/9-1/15)}{3\times 7} + \ldots + u\right)\log N.$$

So by refer to (Hardy and Wright [3]), $\sum_{3\leq q<w}\frac{\log q}{q} < \ \log w + c$, that c is positive constant number, so

$$\log(N-p_r) < 9/4\log 89 + (9/4)c + 0.85\log N.$$

Then for a large N,

$$N < N^{0.9} + p_r,$$

but by our assume $p_r \leq 2N^{2/3}$, so we have

$$N < N^{0.9} + 2N^{2/3},$$

and this is a contradiction for a large N.

References

[1] M. EI Bachraoui, Prime in the interval [$2n$, $3n$], International Journal of Contemporary Mathematical Sciences 1(3) (2006), 617-621.

[2] P. Erdos, Beweis eines satzes von tschebyschef, Acta Litt. Univ. Sci., Szeged, Sect. Math. 5 (1932), 194-198.

[3] G. H. Hardy and E. M. Wright, An Introduction to the Theory of Numbers, Oxford, 1964.

[4] Unsolved Problem in Number Theory, Richard K. Guy, Amazon.com.

[5] P. Erdos and J. Suranyi, Topics in the Theory of Numbers, Undergraduate Texts in Mathematics, Springer Verlag, 2003.

[6] S. Ramanujan, A proof of Bertrand's postulate, Journal of the Indian Mathematical Society 11 (1919), 181-182.

[7] Shiva Kintali, A Generalization of Erdos's Proof of Bertrand-Chebyshev Theorem. http://www.cs.princeton.edu/ kintali, 2008.

[8] H. Sazegar, A method for solving Legendre's conjecture, Journal of Mathematics Research, (2012).

[9] Sinisalo and K. Matti, Checking the Goldbach conjecture up to 4 1011, Mathematics of Computation 61(204) (1993), 931-934. doi:10.2307/2153264.

[10] J. R. Chen, On the representation of a larger even integer as the sum of a prime and the product of at most two primes, Sci. Sinica 16 (1973), 157-176.

[11] D. R. Heath-Brown and J. C. Puchta, Integers represented as a sum of primes and powers of two, Asian Journal of Mathematics 6(3) (2002), 535-565. arXiv:math.NT/0201299.

[12] J. Pintz and I. Z. Ruzsa, On Linnik's approximation to Goldbach's problem, I, Acta Arithmetica 109(2) (2003), 169-194. doi:10.4064/aa109-2-6.

■